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A supersymmetric matrix model: III. Hidden SUSY in statistical systems

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Abstract: The Hamiltonian of a recently proposed supersymmetric matrix model has been shown to become block-diagonal in the large-N, infinite 't Hooft coupling limit. We show that (most of) these blocks can be mapped into seemingly non-supersymmetric (1+1)-dimensional statistical systems, thus implying non-trivial (and apparently yet-unknown) relations within their spectra. Furthermore, the ground states of XXZ-chains with an odd number of sites and asymmetry parameter $\Delta = -1/2$, objects of the much-discussed Razumov-Stroganov conjectures, turn out to be just the strong-coupling supersymmetric vacua of our matrix model.

Keywords: Supersymmetric Effective Theories, 1/N Expansion, Matrix Models, Bethe Ansatz.

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Contents

1. Introduction

In a recent series of papers [1]–[4] we have introduced a supersymmetric quantum mechanical matrix model and studied its rather intriguing properties in the planar approximation, or, formally, in the large-N, fixed-λ limit (here N is the size of our bosonic and fermionic matrices and $\lambda \equiv g^2 N$ is the usual 't Hooft coupling) [5]. In particular, the model exhibits, at a critical value of λ, $\lambda_c = 1$, a discontinuous phase transition characterized by the emergence of new supersymmetric vacua on the strong-coupling side of the phase transition, and a consequent jump of Witten's index [6] across λ_c.

This property was first observed [1, 2] in the lowest fermion-number sector of the model, $F = 0$ (F being an exactly conserved quantum number for all values of λ) where *one* new supersymmetric vacuum emerges at $\lambda > 1$. It was later realized that a similar phenomenon also occurs at $F = 2$ [4], with *two* new supersymmetric vacua popping up at $\lambda > 1$. In that same paper a deeper understanding of this unexpected feature was gained by considering the $\lambda \to \infty$ limit of the model. In this limit the Hamiltonian becomes block-diagonal in both F and boson number B, with blocks of finite size $\mathcal{N}(F, B)$.

Furthermore, by combining the strong-coupling limit with some combinatorics arguments [3], it was conjectured that the pattern found at $F = 2$ should generalize to all even values of F: *two* new supersymmetric vacua would occur in each one of these sectors at large λ. By going to infinite λ, and by computing suitable supertraces [3], we can also guess which blocks should "hide" (for a given even F) the *two* new ground states: those with

$B = F \pm 1$, a conjecture confirmed by many numerical checks. Finally, once the infinite-λ ground states are identified, their expression at finite λ can be reconstructed through an explicit formal operation [4], which is expected to lead to a normalizable state if and only if $\lambda > \lambda_c$.

In this paper we consider again the $\lambda \to \infty$ limit, albeit for a different purpose. We will show that some of the finite blocks (including those where the new supersymmetric vacua occur) can be mapped into (seemingly non-supersymmetric) one-dimensional statistical mechanics models with a finite number of sites and a periodic (cyclic) structure. More explicitly, we will map some sectors of our model into the XXZ Heisenberg chain with asymmetry parameter $\Delta = \pm 1/2$. Moreover, it will be argued, and demonstrated numerically, that our system is also equivalent to a lattice gas of q-bosons in the limit where the quantum-deformation parameter q goes to infinity.

Such kinds of equivalences are not new, the Thirring-sine-Gordon connection [7] being a famous example; however, the equivalence discussed in this paper has two novel features:

- It connects a supersymmetric system with a (seemingly) non-supersymmetric one, hopefully revealing hidden supersymmetric features of the latter model. As an example, the ground state of the XXZ model with $\Delta = -1/2$ is known to have amazing (partly proved, partly conjectured) properties [8] that could possibly be explained after realizing that such a ground state is just a supersymmetric vacuum.

- It relates a rather abstract quantum mechanical matrix model, in the planar approximation, to some well known statistical system in one-space one-time, therefore providing an a priori unexpected physical (string-like?) interpretation of the former.

In the next section we recall the definition of our model, its main physical properties, and its large-λ limit. The equivalence with the XXZ chain and with the q-bosonic gas is discussed in the following two sections. We will end with a summary and a discussion of the possible consequences of this equivalence for the latter two systems.

2. A planar supersymmetric matrix model

The model is simply the $N \to \infty$ (planar) limit of an $N \times N$ matrix system defined by the following supersymmetric charges and Hamiltonian:

$$Q = \mathrm{Tr}[f a^\dagger (1 + g a^\dagger)], \quad Q^\dagger = \mathrm{Tr}[f^\dagger (1 + g a) a], \tag{2.1}$$

$$H = \{Q^\dagger, Q\} = H_B + H_F\,, \tag{2.2}$$

$$H_B = \mathrm{Tr}[a^\dagger a + g({a^\dagger}^2 a + a^\dagger a^2) + g^2 {a^\dagger}^2 a^2]\,, \tag{2.3}$$

$$\begin{aligned} H_F = \ & \mathrm{Tr}[f^\dagger f + g(f^\dagger f (a^\dagger + a) + f^\dagger (a^\dagger + a) f) \\ & + g^2 (f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)]\,, \end{aligned} \tag{2.4}$$

where bosonic and fermionic destruction and creation operators satisfy

$$[a_{ij}, {a^\dagger}_{kl}] = \delta_{il}\delta_{jk} \ ; \ \{f_{ij} f^\dagger_{kl}\} = \delta_{il}\delta_{jk}\,; \ i,j,k,l = 1, \ldots N\,, \tag{2.5}$$

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all other (anti)commutators being zero.

Models of this type result from the dimensional reduction of $D = (1+1)$-dimensional gauge theories. For example, two-dimensional Yang-Mills gluodynamics, when reduced to QM, is described by a free Hamiltonian $H_{\mathrm{YM}_2} = -\mathrm{Tr}[(a - a^\dagger)^2]/2$ acting on the gauge-invariant subspace of the whole Hilbert space [9]. The Hamiltonian (2.2)-(2.4) was designed to illustrate a new, general method [1] of finding the spectrum of gauge systems at infinite number of colours N. It turned out, however, to have an interest of its own, by exhibiting the following non-trivial properties:

- Since (2.4) conserves the fermionic number $F = \mathrm{Tr}[f^\dagger f]$, the system can be studied separately for each F.
- The planar model is exactly soluble in the $F = 0, 1$ sectors, i.e. the complete energy spectrum and the eigenstates are available in analytic form.
- There is a discontinuous phase transition in the 't Hooft coupling at $\lambda = \lambda_c = 1$. At this point the otherwise discrete spectrum loses its energy gap and becomes continuous.
- An exact duality between the strong and weak coupling phases occurs in the $F = 0, 1$ sectors.
- The system exhibits unbroken supersymmetry, i.e. there are exact, SUSY-induced degeneracies between bosonic (even F) and fermionic (odd F) eigenenergies.
- In the weak coupling phase, $\lambda < 1$, there exists only one (unpaired) SUSY vacuum. It lies in the $F = 0$ sector and it is nothing else than the empty Fock state $|0\rangle$. For $\lambda > 1$, however, the structure is much less trivial and more interesting: there are *two* SUSY vacua in each bosonic sector of the model. For $F = 0$ the empty Fock state continues to be a null eigenstate, but it is joined by another, non-trivial, analytically known ground state. For higher even F, two new non-trivial vacua appear. This is possible thanks to the rather intriguing rearrangement of the members of supermultiplets that occurs across the phase-transition point.

We have established all these points for the $F = 0, 1, 2, 3$ sectors and believe that this structure persists for arbitrary F. This expectation is borne out by considering the infinite $\lambda \to \infty$ limit of the Hamiltonian (2.4). Since this is also the limit in which our Hamiltonian reveals the above-mentioned connections to statistical mechanics, let us recall the strong coupling limit of our system [4] in a little more detail.

Define the (appropriately rescaled) strong coupling SUSY charges by:

$$Q_{\mathrm{SC}} = \lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}}\, Q = \frac{1}{\sqrt{N}} \mathrm{Tr}(f a^{\dagger 2}) \ , \quad Q^\dagger_{\mathrm{SC}} = \frac{1}{\sqrt{N}} \mathrm{Tr}(f^\dagger a^2)\,. \tag{2.6}$$

The corresponding strong-coupling Hamiltonian is just their anticommutator. Doing the algebra carefully, and throwing away terms that do not contribute in the large-N limit, we find:

$$H_{\rm SC} = \lim_{\lambda\to\infty} \frac{1}{\lambda} H = \frac{1}{N} {\rm Tr}[a^{\dagger 2} a^2 + f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a] \,. \tag{2.7}$$

This can be further simplified with the aid of the "planar calculus" rules derived in [1, 4]; namely, the third and the fourth terms must be brought to normal form, giving:

$$f^\dagger_{ij} a_{jk} {a^\dagger}_{kl} f_{li} \;=\; f^\dagger_{ij} ({a^\dagger}_{kl} a_{jk} + \delta_{jl}\delta_{kk}) f_{li} \to N {\rm Tr}[f^\dagger f], \tag{2.8}$$

$$f^\dagger_{ij} f_{jk} {a^\dagger}_{kl} a_{li} \;=\; f^\dagger_{ij} {a^\dagger}_{kl} f_{jk} a_{li} \to 0 \,. \tag{2.9}$$

In the above relations we have neglected terms in which the normal ordering does not match the trace-ordering, since such structures do not contribute to leading order in $1/N$ [1, 4]. In conclusion, the strong-coupling Hamiltonian reads:[1]

$$H_{\rm SC} = {\rm Tr}[f^\dagger f + \frac{1}{N}(a^{\dagger 2} a^2 + a^\dagger f^\dagger a f + f^\dagger a^\dagger f a)]. \tag{2.10}$$

Remarkably, (2.10) conserves *both* F *and* $B = {\rm Tr}[a^\dagger a]$. As such, the infinite Hamiltonian matrix splits into *finite* blocks labelled by the number of fermionic and bosonic quanta, (F, B). In each such sector the leading-order (i.e. planar) basis is generated by the single trace of a product of elementary creation operators. We may thus represent the generic state in a block of given B and F in the form:

$$|m_i, n_i\rangle = \frac{1}{\mathcal{N}_n} {\rm Tr}[a^{\dagger m_1} (f^\dagger)^{n_1} a^{\dagger m_2} (f^\dagger)^{n_2} \ldots (f^\dagger)^{n_k}] |0\rangle \,;\; m_i, n_i > 0 \,. \tag{2.11}$$

This is a state with $B = \sum m_i$ bosons and $F = \sum n_i$ fermions, $\mathcal{N}_n$ being a normalization factor. Owing to the Pauli principle not all configurations of $\{m_i, n_i\}$ define a legitimate state. The detailed rules for counting such states (Pauli-allowed necklaces) follow from Polya's theory and have been developed in [3]: they give the dimensionality of each block of the strong-coupling Hamiltonian (2.10). Table 1 shows a map of the first few (F, B) sectors together with their sizes. Since, in the strong-coupling limit, the supersymmetric charges connect states with the same value of $B+2F$, taking supertraces at fixed $B+2F$ is a way to check whether all states with such a value of $B+2F$ are paired into supermultiplets or not. Such an analysis was carried out in [3] and revealed a bosonic excess by one unit for (and only for) $B + 2F = \pm 1 (\mathrm{mod}\ 6)$.

Using the planar rules developed in [1, 2, 4] the finite-dimensional Hamiltonian matrix can be readily calculated in each sector. Proceeding in this way, we have discovered "experimentally" the existence of a "magic staircase" – a distinct set of sectors (labelled by a bold face in table 1) – where the zero-energy eigenstates are located.[2] Indeed the magic sectors appear to lie at

$$B = F \pm 1 \;,\; F = 2n \;\Rightarrow B + 2F = 3F \pm 1 = 6n \pm 1 \,, \tag{2.12}$$

[1]To be precise $H_{\rm SC}$ contains also a ${\rm Tr}(f^\dagger){\rm Tr}(f)$ term. It is subleading *except* when it acts on $|B = 0, F = 1\rangle$ and plays an important role for assuring degeneracy with $|B = 2, F = 0\rangle$.

[2]The dimensionality of these sectors turns out to be given by Catalan's numbers [3].

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B \ F	0	1	2	3	4	5	6	7	8	9	10
11	1	1	6	26	91	273	728	1768	3978	8398	**16796**
10	1	1	5	22	73	201	497	1144	2438	4862	9226
9	1	1	5	19	55	143	335	715	**1430**	2704	**4862**
8	1	1	4	15	42	99	212	429	809	1430	2424
7	1	1	4	12	30	66	**132**	247	**429**	715	1144
6	1	1	3	10	22	42	76	132	217	335	497
5	1	1	3	7	**14**	26	**42**	66	99	143	201
4	1	1	2	5	9	14	20	30	43	55	70
3	1	1	**2**	4	**5**	7	10	12	15	19	22
2	1	1	1	2	3	3	3	4	5	5	5
1	**1**	1	**1**	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	1	0	1	0

Table 1: Sizes of gauge-invariant bases in the (F, B) sectors

in agreement with the supertrace considerations of [3]. This observation also explains the structure of SUSY vacua in the whole strong-coupling phase. Indeed, we were able [4] to give a (formal) way to express SUSY vacua at finite λ in terms of those at $\lambda = \infty$. The connection is expected to lead to normalizable states only at $\lambda > 1$.

It remains, however, to understand *why* the strong SUSY vacua exist solely in the magic sectors (2.12). This puzzle finds its solution upon mapping our system into the XXZ Heisenberg chain.

3. Equivalence with the XXZ Heisenberg chain

3.1 The XXZ model

The XXZ system [10−12] is a one-dimensional periodic lattice of size L with a spin 1/2 variable residing on its sites. Its Hamiltonian depends on an "asymmetry" parameter Δ and can be written in terms of Pauli's matrices (with $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$) as:

$$\begin{aligned} H^{(\Delta)}_{\rm XXZ} &= -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma^x_i\sigma^x_{i+1} + \sigma^y_i\sigma^y_{i+1} + \Delta\ \sigma^z_i\sigma^z_{i+1}\right) \\ &= -\sum_{i=1}^{L}\left(\sigma^+_i\sigma^-_{i+1} + \sigma^-_i\sigma^+_{i+1} + \frac{\Delta}{2}\sigma^z_i\sigma^z_{i+1}\right) \equiv -\left(O_\pm + \frac{\Delta}{2}O_z\right), \end{aligned} \tag{3.1}$$

where the site $L+1$ is identified with the site 1. A convenient basis of states, labelled by two sets of integers (m_i, n_i), is the following:

$$|m_i, n_i\rangle \equiv |(0)^{m_1}(1)^{n_1}\dots(0)^{m_r}(1)^{n_r}\rangle\ ,\ r \geq 1, m_i, n_i > 0\,,\ \sum(m_i + n_i) = L\,, \tag{3.2}$$

where a $(0)^m$ (resp. a $(1)^n$) indicates a sequence of m (resp. n) spins pointing down (resp. up). The state is defined up to cyclic permutations and the permutation giving the smallest

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binary number can be taken as representative. To the set (3.2) one still has to add two states: $(0)^L$ and $(1)^L$.

The action of O_z on such states is very simple:

$$O_z \, |m_i, n_i\rangle = (n_{\rm eq} - n_{\rm opp}) \;\; |m_i, n_i\rangle \; , \tag{3.3}$$

where $n_{\rm eq}$ ($n_{\rm opp}$) is the number of equal(opposite)-spin nearest neighbours. It is easy to see that $n_{\rm eq} = L - 2r$ while $n_{\rm opp} = 2r$. Therefore:

$$O_z \, |m_i, n_i\rangle = (L - 4r) \;\; |m_i, n_i\rangle \; . \tag{3.4}$$

The action of $O_\pm$ is only a bit more complicated:

$$O_\pm \, |m_i, n_i\rangle = \sum \left|m'_j, n'_j\right\rangle \; , \tag{3.5}$$

where the sum extends over the sets (m'_j, n'_j) that are obtained from (m_i, n_i) by interchanging, in turn one by one, any pair of opposite-spin neighbours. As a result, the Hamiltonian commutes with the z-component of the total spin and is block-diagonal with blocks of given $m = \sum m_i$ and $n = \sum n_i$ $(n + m = L)$. In conclusion:

$$\begin{aligned} H^{(\Delta)}_{\rm XXZ} \;\; |m_i, n_i\rangle &= -\frac{\Delta}{2}(L - 4r) \;\; |m_i, n_i\rangle - \sum \left|m'_j, n'_j\right\rangle \\ &= \left(\frac{3}{2}\Delta \; L - 2\Delta \; (L - r)\right) |m_i, n_i\rangle - \sum \left|m'_j, n'_j\right\rangle \; . \end{aligned} \tag{3.6}$$

It is useful to define a rescaled XXZ Hamiltonian by:

$$\begin{aligned} \tilde{H}^{(\Delta)}_{\rm XXZ} &\equiv -\frac{1}{2\Delta} H^{(\Delta)}_{\rm XXZ} = \frac{1}{4} O_z + \frac{1}{2\Delta} O_\pm \\ &= \frac{1}{4} \sum_{i=1}^{L} \left(\sigma^z_i \sigma^z_{i+1} + \frac{2}{\Delta} (\sigma^+_i \sigma^-_{i+1} + \sigma^-_i \sigma^+_{i+1}) \right) \; , \end{aligned} \tag{3.7}$$

so that:

$$\tilde{H}^{(\Delta)}_{\rm XXZ} \;\; |m_i, n_i\rangle = \left(-\frac{3}{4} \; L + (L - r) \right) |m_i, n_i\rangle + \frac{1}{2\Delta} \sum \left|m'_j, n'_j\right\rangle \; . \tag{3.8}$$

For the special values of the asymmetry parameter $\Delta = \pm\frac{1}{2}$ we get:

$$\tilde{H}^{(\pm 1/2)}_{\rm XXZ} = -\frac{3}{4} L + H^{(\pm)} \, , \tag{3.9}$$

where:[3]

$$H^{(\pm)} \;\; |m_i, n_i\rangle = (L - r) \, |m_i, n_i\rangle \pm \sum \left|m'_j, n'_j\right\rangle \; . \tag{3.10}$$

We will now argue that the strong-coupling Hamiltonian of our supersymmetric matrix model can be mapped into $H^{(\pm)}$ for some specific values of B, F that include those of the magic staircase.

[3] We are grateful to Don Zagier for having pointed out to us this simple version of the XXZ Hamiltonian and for having suggested a possible relation to our matrix model.

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3.2 Proof of the equivalence

Let us start by splitting the strong-coupling Hamiltonian (2.10) as follows:

$$H_{\rm SC} = H_{\rm SC}^{(d)} + H_{\rm SC}^{(o)}\,, \tag{3.11}$$

where:

$$H_{\rm SC}^{(d)} = {\rm Tr}(f^\dagger f) + \frac{1}{N}{\rm Tr}({a^\dagger}^2 a^2)\,, \tag{3.12}$$

$$H_{\rm SC}^{(o)} = \frac{1}{N}\left[f^\dagger a^\dagger f a + a^\dagger f^\dagger a f\right]\,, \tag{3.13}$$

and the labels d (o) stand for diagonal (off-diagonal) pieces of the Hamiltonian.

Let us now consider the action of $H_{\rm SC}$ on the generic state (2.11) whose similarity with the XXZ states (3.2) is evident. The planar rules for doing that were discussed in [1, 2]. For $H_{\rm SC}^{(d)}$ we simply find ($F = n$):

$$H_{\rm SC}^{(d)}|n_i, m_i\rangle = \left(F + \sum_{i=1}^{r}(m_i - 1)\right)|n_i, m_i\rangle = (L - r)\;|n_i, m_i\rangle\,. \tag{3.14}$$

The action of $H_{\rm SC}^{(o)}$, on the other hand, is precisely to interchange each fermion-boson pair of neighbours, i.e. an action very close to that of $O_\pm$ in the XXZ model. However, since sign problems can arise, we have to treat various cases separately:

- F odd

  In this case there are no relative signs from different cyclic orderings in (2.11) and therefore the action of $H_{\rm SC}^{(o)}$ on them is exactly the same as that of $O_\pm$ on the states (3.2). The relative sign of $H_{\rm SC}^{(d)}$ and $H_{\rm SC}^{(o)}$ in $H_{\rm SC}$ concides with the one in $H^{(+)}$. We thus obtain:

  $$H_{\rm SC} \Leftrightarrow H^{(+)} = \tilde{H}_{\rm XXZ}^{(+1/2)} + \frac{3}{4}L = -H_{\rm XXZ}^{(+1/2)} + \frac{3}{4}L\,. \tag{3.15}$$

- F even and B odd

  In this case different cyclic orderings in (2.11) do carry relative signs. We will argue that, by a suitable choice of the basis, we can bring $H_{\rm SC}$ to agree, up to a shift, with $H_{\rm XXZ}^{(-1/2)}$:

  $$H_{\rm SC} \Leftrightarrow H^{(-)} = H_{\rm XXZ}^{(-1/2)} + \frac{3}{4}L\,. \tag{3.16}$$

  The argument goes as follows: for the XXZ model let us choose as basis of states:

  $$(1111\ldots1100\ldots000), (1111\ldots1010\ldots000), \ldots \tag{3.17}$$

  The corresponding basis for $H_{\rm SC}$ is then taken to be:

  $$|f^\dagger f^\dagger f^\dagger f^\dagger\ldots f^\dagger f^\dagger a^\dagger a^\dagger\ldots a^\dagger a^\dagger a^\dagger), -|f^\dagger f^\dagger f^\dagger f^\dagger\ldots f^\dagger a^\dagger f^\dagger a^\dagger\ldots a^\dagger a^\dagger a^\dagger\rangle, \ldots\,, \tag{3.18}$$

i.e. we put in correspondence the ones (zeroes) in (3.17) with the fermions (bosons) in (3.18), while assigning a sign $(-1)^k$ to a state in the latter set if it is obtained from the first state by interchanging k boson-fermion pairs.

It is quite obvious that, on this convenient basis, $H_{\rm SC}$ gives the same matrix elements as $H^{(-)}$, modulo the possibility that $H_{\rm SC}$ produces a cyclic permutation of a state in the above basis. However, even in this case, the correspondence works fine thanks to the fact that the relative sign $(-1)^p$ originating from Fermi statistics (where p is the number of fermions to be interchanged) is equal to the relative sign $(-1)^k$ counting the number of boson-fermion interchanges. This is so since taking a fermion from the last to the first entry corresponds to an odd number of fermion-fermion *and* boson-fermion interchanges, while doing the same with a boson involves an even number of interchanges of each type. Note that for this to be true it is essential that B be odd and F be even, and, therefore, that L be odd. The importance of L being odd was much emphasized in [8].

- F and B even

  In this case we have found no simple relation between $H_{\rm SC}$ and $H_{\rm XXZ}$ for any choice of Δ.

- We may add here a side remark: in the case of F odd, if we define a new (non-supersymmetric) Hamiltonian:

$$\tilde{H}_{\rm SC} = H_{\rm SC}^{(d)} - H_{\rm SC}^{(o)}\,, \tag{3.19}$$

  we also find:

$$\tilde{H}_{\rm SC} \Leftrightarrow H^{(-)} = H_{\rm XXZ}^{(-1/2)} + \frac{3}{4}L\,. \tag{3.20}$$

3.3 Implications of supersymmetry on the XXZ model

The final form of our equivalences reads:

$$H_{\rm SC}(F,B) = \begin{cases} -H_{\rm XXZ}^{(+1/2)} + \frac{3}{4}L\,, & F \text{ odd}, \\ +H_{\rm XXZ}^{(-1/2)} + \frac{3}{4}L\,, & F \text{ even}, B \text{ odd}\,. \end{cases} \tag{3.21}$$

Eq. (3.2) implies that the parameters of both systems are related as follows:

$$L = F + B, \tag{3.22}$$

$$S_z = \frac{1}{2}(F-B)\,, \tag{3.23}$$

where S_z is the conserved component of the total spin. In addition, the spectrum on the spin side should be computed in the sector that is invariant under the lattice shifts.

We checked eqs. (3.21) for all sectors with $5 \leq F+B \leq 9$. Everything works as expected, including the magic sectors with a zero eigenvalue. Since the spectrum of $H_{\rm SC}$ is positive semi-definite, the 2nd of eqs. (3.21) gives a simple and elegant proof that, for L odd, the states considered in [8] are indeed ground states of the XXZ chain. The fact that

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they have $S_z = \pm\frac{1}{2}$ just corresponds to our SUSY ground states having $B = F \pm 1$ and F even. Moreover, for sectors with F *and* B even, where the equivalence is not expected to work, we indeed find different spectra in the two models.

Supersymmetry also implies that the (non-vanishing) spectrum of $H_{\rm SC}$ in the sector (F, B) has to be contained in the spectra of the "neighbouring" sectors with $(F \mp 1, B \pm 2)$. If we take B odd, $B \pm 2$ is also odd and therefore we are always in a situation in which we are able to connect $H_{\rm SC}(F, B)$ to $H_{\rm XXZ}(n, m)$. Obviously, $n = F$ and $m = B$ are conserved by $H_{\rm XXZ}$ as well as by $H_{\rm SC}$. After some trivial algebra, we get the following predictions (with integer μ and ν ensuring even/odd m and n respectively):

1. The spectrum of $H^{(+1/2)}_{\rm XXZ}(2\nu + 1, 2\mu + 1)$ is contained in the combined spectra of $-H^{(-1/2)}_{\rm XXZ}(2\nu, 2\mu + 3) - 3/4$ and $-H^{(-1/2)}_{\rm XXZ}(2\nu + 2, 2\mu - 1) + 3/4$,

   and:

2. The excited spectrum of $H^{(-1/2)}_{\rm XXZ}(2\nu, 2\mu + 1)$ is contained in the combined spectra of $-H^{(+1/2)}_{\rm XXZ}(2\nu - 1, 2\mu + 3) + 3/4$ and $-H^{(+1/2)}_{\rm XXZ}(2\nu + 1, 2\mu - 1) - 3/4$.

Since the $H_{\rm XXZ}(n, m)$ Hamiltonian is symmetric under the exchange of m and n, all variations of the above relations resulting from the interchange $m \leftrightarrow n$ are also valid. We made extensive numerical checks of these (to our knowledge novel) relations between different XXZ models at different values of Δ.

3.4 Bethe ansatz solutions for the lower sectors of the XXZ chain

The XXZ chain is integrable [12]. In particular the eigenenergies of $H_{\rm XXZ}(\Delta)$ are given exactly by the Bethe ansatz [10, 13]:

$$E_{\rm XXZ}(\Delta) = -L\frac{\Delta}{2} + 2m\Delta - 2\sum_{j=1}^{m} \cos p_j, \tag{3.24}$$

where the momenta $-\pi < p_j < \pi$ satisfy the following set of Bethe equations:

$$e^{iLp_j} = (-1)^{m-1} \prod_{l=1, l\neq j}^{m} e^{i(p_j - p_l)} \frac{e^{ip_l} + e^{-ip_j} - 2\Delta}{e^{ip_j} + e^{-ip_l} - 2\Delta}, \quad j = 1, \ldots, m. \tag{3.25}$$

With m denoting the number of down spins in a chain. For the supersymmetric model this translates into

$$E_{\rm SC} = F + 2\textstyle\sum_{j=1}^{B} \cos p_j, \quad \text{for } F \text{ odd}, \ \Delta = +\frac{1}{2} \tag{3.26}$$

$$E_{\rm SC} = F - 2\textstyle\sum_{j=1}^{B} \cos p_j, \quad \text{for } F \text{ even, and } B \text{ odd}, \ \Delta = -\frac{1}{2}. \tag{3.27}$$

Given the Bethe momenta p_i, all eigenvectors can also be explicitly constructed. The whole problem reduces therefore to the solution of the non-linear equations (3.25). Consequently, the existence of the magic staircase with its supersymmetric vacua follows directly from

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(F,B)	$\mathcal{N}(F,B)$	p_1/π	p_2/π	p_4/π	p_6/π
$(4,3)$	5	0.0	1/3		
$(4,5)$	14	0.0	0.260669	0.558585	
$(6,5)$	42	0.0	0.210767	0.432222	
$(6,7)$	132	0.0	0.178899	0.364275	0.583645

Table 2: Bethe momenta of the supersymmetric vacua in some of the lowest magic (F,B) sectors, see table 1

the Bethe ansatz solution of the XXZ chain. The literature on the latter subject is huge, see e.g. [12]–[16] for more references. We shall content ourselves here with formulating only an "ansatz within the Bethe ansatz", which reduces the number of Bethe momenta needed to find our SUSY vacua. Solving numerically eq. (3.25) for a few low-F magic sectors, we have found that the zero-energy momenta satisfy (see table 2):

$$p_1 = 0, \quad p_{2k+1} = -p_{2k}, \quad k = 1, \ldots, (B-1)/2. \tag{3.28}$$

In words, there is always one zero momentum, and the remaining ones come in pairs with opposite sign.[4] We conjecture that this configuration gives the zero-energy state for arbitrary $B = F \pm 1$ and even F.

It is perhaps amusing that the ansatz (3.28) allows the Bethe phase factors to be derived for the first three vacua listed in table 2 in analytic form. Assuming that they correspond to zero-energy eigenstates, we look for simultaneous solutions of eqs. (3.25) and $E(F,B) = 0$. This problem can be solved algebraically. Defining $x_i = e^{ip_i}$, we find:

$$\begin{aligned}
&\boldsymbol{F = 4, \quad B = 3:}\\
x_2 &= \frac{1}{2}(1 + i\sqrt{3})\\
&\boldsymbol{F = 4, \quad B = 5:}\\
x_2 &= \frac{1}{64}\Bigg(16 - i\sqrt{2}\sqrt{15+\sqrt{33}}(7-\sqrt{33})\\
&\qquad +4\sqrt{-16(3+\sqrt{33}) + i2\sqrt{2}\sqrt{15+\sqrt{33}}(9+\sqrt{33})}\Bigg)\\
x_4 &= \frac{1}{64}\Bigg(16 + i\sqrt{2}\sqrt{15+\sqrt{33}}(7-\sqrt{33})\\
&\qquad -4\sqrt{-16(3+\sqrt{33}) - i2\sqrt{2}\sqrt{15+\sqrt{33}}(9+\sqrt{33})}\Bigg)
\end{aligned}$$

[4] Recall that the magic sectors only occur for odd values of B.

$F = 6, \ \ B = 5:$

$$x_2 = \frac{1}{72}\Bigg(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13})$$
$$+6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\Bigg)$$
$$x_4 = \frac{1}{72}\Bigg(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13})$$
$$-6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\Bigg),$$

corresponding to the algebraic representations of the phase factors given in table 2. Although it is not self-evident, it can be proved algebraically that the above numbers have modulus 1.

3.5 The Razumov-Stroganov states and supersymmetry

Interestingly, the magic staircase, with its supersymmetric vacua, connects directly to properties of one-dimensional spin chains. Beginning with the classic paper of Baxter [11], some simple eigenvalues of the XXZ Heisenberg chain were discovered. In particular, Baxter has shown the existence of a ground state with $S_z = \pm\frac{1}{2}$ and energy $-\frac{3}{4}L$ for infinite L and $\Delta = -\frac{1}{2}$. More recently, Razumov and Stroganov [8] have extended this result to any finite, odd L and have made several conjectures on the properties of the eigenvector that corresponds to the above-mentioned ground state (for recent developments see, e.g. [15]).

It follows directly from (3.21) and (3.23) that these states are nothing but the supersymmetric vacua of our planar model with $B = F \pm 1, F$ *even*. Hence Razumov and Stroganov's above-mentioned result guarantees the existence of one bosonic SUSY vacuum in each one of our magic sectors. Hopefully, this hidden supersymmetry will help understanding and/or proving the other amazing (and so far mostly conjectured by RS) properties of Baxter's ground states.

It is tempting to say that the two families of vacua, i.e. those with some given even F and $B = F \pm 1$, are related by the usual inversion, $\sigma_i \to -\sigma_i$, symmetry. However, this transformation has to be *coupled* with a change in the lattice size $L \to L+2$. This brings the issue of whether a change of L should be considered as a symmetry. At first sight this looks a little premature, and in fact, for the above vacuum solutions, it is pure semantics. However, this is no longer the case when we consider the implications of supersymmetry on the whole spectrum, including all excited states. The supersymmery generators act as $F \to F \pm 1$ and $B \to B \mp 2$, corresponding to $L \to L \mp 1$. Therefore they do relate the spectra of excited states on *different* lattices: the XXZ chain turns out to have a hidden supersymmetry with different members of its dynamical supermultiplets living on different lattices!

4. Equivalence with a q-bosonic gas

Surprisingly, our infinite-coupling planar system is exactly equivalent to yet another, and apparently entirely different, model. To expose this equivalence we use the original labelling of planar states with F fermions

$$|n_1, n_2, \ldots, n_F\rangle = \frac{1}{\mathcal{N}_n} \mathrm{Tr}[a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger \ldots a^{\dagger n_F} f^\dagger]|0\rangle \; . \tag{4.1}$$

Consider now the action of the strong-coupling Hamiltonian,

$$H_{\mathrm{SC}} = \mathrm{Tr}[f^\dagger f + \frac{1}{N}(a^{\dagger 2} a^2 + a^\dagger f^\dagger a f + f^\dagger a^\dagger f a)] \, , \tag{4.2}$$

on a state (4.1). Using the planar rules developed in [1]–[4], it is easy to show that the first two terms do not change the initial state and give rise to the following diagonal elements:

$$\langle n_1, n_2, \ldots, n_F | H_{\mathrm{SC}} | n_1, n_2, \ldots, n_F\rangle = F + \sum_{i=1}^{F} (n_i - 1 + \delta_{n_i,0}) = B + \sum_{i=1}^{F} \delta_{n_i,0} \, . \tag{4.3}$$

Using the same planar rules, the remaining matrix elements can be easily obtained from the action of the last two terms, e.g.

$$\begin{aligned}\mathrm{Tr}[a^\dagger f^\dagger a f]|n_1, n_2, \ldots, n_F\rangle = {} & \frac{\mathcal{N}_{n_1}}{\mathcal{N}_n}|n_1 - 1, n_2, \ldots, n_F + 1\rangle + \frac{\mathcal{N}_{n_2}}{\mathcal{N}_n}|n_1 + 1, n_2 - 1, \ldots, n_F\rangle + \\ & \frac{\mathcal{N}_{n_3}}{\mathcal{N}_n}|n_1, n_2 + 1, n_3 - 1, \ldots, n_F\rangle + \ldots + \frac{\mathcal{N}_{n_F}}{\mathcal{N}_n}|n_1, n_2, \ldots, n_{F-1} + 1, n_F - 1\rangle \, ,\end{aligned} \tag{4.4}$$

i.e. this operator annihilates one bosonic quantum in a group i and adds one at $i-1$, meaning at the cyclic left of $i, i = 1, \ldots, F$. Similarly the last term moves one quantum to the cyclic right of i. Here, $\mathcal{N}_n$ ($\mathcal{N}_{n_f}$) are the normalization factors of the initial (final) states of our basis. They contain some powers of N and degeneracy factors d_s. When calculating matrix elements of the Hamiltonian H_{SC}, all N factors cancel with the $1/N$ in (4.2) and we are left only with the square roots of the ratios of corresponding d_s factors.

It turns out that this Hamiltonian also describes the following system. Consider a one-dimensional, periodic lattice of length F. Put at each lattice site a bosonic degree of freedom described by the creation/annihilation (c/a) operators a_i, $i = 1, \ldots, F$ and use the harmonic oscillator basis. The states $|n_1, n_2, \ldots, n_F\rangle$ are described by the configuration of F integer occupation numbers as before.

The new Hamiltonian reads

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b_{i+1}^\dagger + b_i b_{i-1}^\dagger, \tag{4.5}$$

where N_i is the usual operator of the number of quanta at a site i, $N_i = {a^\dagger}_i a_i$. In words: the second term counts the number n_{zer} of empty (unoccupied) sites in a given basis state and returns this state multiplied by n_{zer}. B is the total number of bosonic quanta,

$B = n_1 + n_2 + \ldots + n_F$. The $b_i^\dagger$ (b_i) operators create (annihilate) one quantum *without* the usual $\sqrt{n}$ factors. Omitting momentarily the index i:

$$b^\dagger|n\rangle = |n+1\rangle, \quad b|n\rangle = |n-1\rangle, \quad b|0\rangle \equiv 0\,. \tag{4.6}$$

In terms of the usual $a, a^\dagger$ operators they read:

$$b^\dagger = a^\dagger \frac{1}{\sqrt{N+1}}, \qquad b = \frac{1}{\sqrt{N+1}}a, \quad \text{and} \quad b|0\rangle \equiv 0\,, \tag{4.7}$$

where again $N = a^\dagger a$. The b operators have non-standard commutation relations:

$$[b, b^\dagger] = \delta_{N,0}\,. \tag{4.8}$$

This Hamiltonian conserves the total bosonic number, as before. It is also invariant under lattice shifts and, consequently, commutes with the lattice-shift operator U defined as:

$$U|n_1, n_2, \ldots, n_F\rangle = |n_2, n_3, \ldots, n_F, n_1\,.\rangle \tag{4.9}$$

Therefore, the Hilbert space of states with fixed B can be further split into sectors with fixed eigenvalues of U:

$$\lambda_U^{(m)} = e^{im\frac{2\pi}{F}}, \quad m = 1, 2, \ldots, F\,. \tag{4.10}$$

We claim that the spectrum of the above H, in the sector with $\lambda_U = (-1)^{F-1}$, exactly coincides with our spectrum of $H_{\rm SC}$, *for all* F and B.

The main steps in understanding this equivalence are as follows:

- Our planar states (4.1) are defined modulo Z_F shifts. Without fermionic degrees of freedom this would be taken care of by requiring $\lambda_U = 1$ for the bosonic system. The minus sign is the consequence of the fermionic operators in (4.1): under the Z_F shifts they acquire the phase $(-1)^{F-1}$.

- For even F, the projection for the $\lambda_U = -1$ sector plays another important role. Namely, it removes states that are not allowed by the Pauli principle.

- Finally, the degeneracy factors required in (4.4) are correctly taken into account by the linear combinations corresponding to the condition $\lambda_U = (-1)^{F-1}$.

The last point is particularly non-trivial. We have therefore double-checked this equivalence by diagonalizing both Hamiltonians in a range of sectors: $3 \le F \le 7, \quad 3 \le B \le 7$. All spectra are indeed identical, including again the supersymmetric vacua in the magic sectors.

Notice that this second equivalence also works for the "bad" sectors with both F and B even. These are the only sectors where the Pauli principle is effective and eliminates some of the planar states. This is why there is no XXZ equivalence in these sectors. Nevertheless, because of the $\lambda_U = -1$ projection, the bosonic-model equivalence applies to these cases as well.

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Interestingly, the system of "funny" b and $b^\dagger$ operators turns out to be a particular limit of the so-called q-deformed harmonic oscillator algebra, well known in the literature [17, 18]. The transitions (4.6) (without the square roots) are referred to as *assisted* transitions.

The q-boson operators satisfy (for one degree of freedom)

$$bb^\dagger - q^{-2}b^\dagger b = 1, \quad [N, b] = -b, \quad [N, b^\dagger] = b^\dagger\,, \tag{4.11}$$

with $N = a^\dagger a$, the usual occupation number operator. The $b, b^\dagger$ operators are related to the standard $a, a^\dagger$ by

$$b = \sqrt{\frac{[N+1]_q}{N+1}}a, \quad b^\dagger = a^\dagger\sqrt{\frac{[N+1]_q}{N+1}}, \tag{4.12}$$

where:

$$[x]_q \equiv \frac{1-q^{-2x}}{1-q^{-2}}\,. \tag{4.13}$$

An alternative form of the commutation relations reads:

$$[b, b^\dagger] = q^{-2N}\,. \tag{4.14}$$

It can readily be checked that the usual harmonic oscillator algebra is recovered in the limit $q \to 1$, $b, b^\dagger \to a, a^\dagger$. On the other hand, in the limit $q \to \infty$, b and $b^\dagger$ become our c/a operators satisfying (4.6)–(4.8).

An important point is that the q-bosonic Hamiltonian without the commutator (or δ) term in (4.5) is exactly soluble for all values of the deformation parameter q. However, with the additional δ term, it is not. On the other hand, given the present chain of equivalences, we see that the above non-linear system of $q = \infty$-bosons *is* soluble in terms of the Bethe ansatz for the XXZ chain. Finally, and similarly to the latter case, the equivalence we observed exposes a hidden, unbroken supersymmetry of ∞-bosons with supersymmetric partners living on lattices of different sizes.

5. Discussion

This article is the third in a series studying the quantum mechanics of a simple supersymmetric matrix model. Designed originally to illustrate the usefulness of the large-N approximation directly in terms of a Hamiltonian and a Hilbert space, the model turned out to have a very rich physics by itself, as amply illustrated in [1]–[4].

Here we have uncovered an even more intriguing aspect of this model: its connection, in the infinite 't Hooft-coupling limit, to one-dimensional statistical systems in which supersymmetry, if present, is very well concealed. We have found that our supersymmetric planar model is exactly equivalent to *two* such systems: the quantum XXZ Heisenberg chain, and a lattice gas of q-bosons.

The intriguing pattern of strong coupling vacua discovered in the matrix model finds its explanation in the unusually simple ground states of the XXZ chain found by Baxter more than thirty years ago. Vice versa, the XXZ chain turns out to have a hidden supersymmetry, which explains a host of degeneracies between seemingly unrelated energy eigenstates. Interestingly, the supersymmetry transformations change the number of lattice sites, so that different members of the supermultiplets live on different lattices. But even within a single sector/lattice, supersymmetry may turn out to be a powerful tool for studying the properties of the ground state and for understanding the meaning of the RS conjectures [8]. The fact that such ground states are annihilated by two supercharge operators should imply distinct properties for them, while finding an operator with the right algebra would allow to generate the full "staircase" of ground states starting from the lowest and simplest ones.

The second system, a gas of q-bosons in the limit of an infinite deformation parameter q, is equivalent to our matrix model to an even greater degree than the Heisenberg chain. While the equivalence with the XXZ model holds only for a subset of all sectors of the Hilbert space, the ∞-bosonic gas is equivalent in all sectors of conserved boson number and for all lattice sizes. To the best of our knowledge, the specific Hamiltonian of the bosonic gas was unsolved till now; but in view of our chain of equivalences, the system should turn out to be actually soluble (e.g. via the Bethe ansatz for the XXZ chain).

Finally, exact solubility of the XXZ model directly implies that same property for our large-N, supersymmetric matrix model at infinite 't Hooft coupling and, therefore, in "three quarters" of its fermionic sectors (since we have found no correspondence for the even-B, even-F sectors). There are also indications that this solubility can be extended to the whole strong-coupling phase using the technique developed in [4]. The reverse is also an interesting question: are there statistical systems that can be mapped into our matrix model at *finite* 't Hooft coupling?

Recently relations between the field theories and spin chains have become the subject of much interest and excitement in connection with the AdS/CFT correspondence, which opens a new way of studying gauge theories [19]-[24].[5] In particular, the mapping between the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and the XXZ chain with $\Delta = 1/2$ has been discussed by Belitsky et al. [20]. It is perhaps not an accident that we find a similar gauge-spin relation in our much simpler model.

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[5] We recall that, so far, the easiest applications of the AdS/CFT correspondence are also made in the $\lambda \to \infty$ (i.e. supergravity) limit.

Note added in proofs

After this paper had been submitted we were informed by Jan de Gier that a "hidden" supersymmetry of the XXZ spin chain had already been pointed out in ref. [25] and further studied in later work (e.g. in [26]). It looks however that, while in those papers supersymmetry is realized non-linearly just in terms of fermionic variables, in our case it takes a simpler form directly in terms of an equal number of elementary bosonic and fermionic quanta. We are grateful to Dr. de Gier for this information.

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